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Analysis of Massless Elastic Chains With Servo Controlled Joints[1]

The lumping approximation used frequently for dynamic analysis of distributed parameter systems is facilitated for a class of flexible systems by a technique using 4×4 coordinate transformation matrices to account for the deflection of elastic elements under load. This approach is used to develop the linear equations of spatial motion for a system of two rigid masses connected by a chain with an arbitrary number of massless beams and controlled joint rotations.

Introduction

A popular approximation used when obtaining models of mechanical systems with both inertia and compliance is the lumped parameter approximation. Mass and rotary moment of inertia effects of a distributed parameter system are lumped together as are compliance effects to obtain an ordinary differential equation for the system. This paper develops a method of implementing these approximations when the compliance is due to a spatial chain of elastic elements. The angular orientation of the joints may be controlled by a linear feedback control system. Obtaining the equations of motion for the system becomes a complex bookkeeping problem which is efficiently handled by 4×4 transformation matrices.

The lumping approximation is widely used in all engineering fields. The current and future requirements of the space program for light manipulator arms and cranes to handle massive objects in zero gravity has given the impetus for developing this more efficient and economical method of obtaining accurate dynamic equations with ease.

Literature on dynamic analysis for manipulator arms includes two approaches by this author [1, 2] and work by others. Impedance methods for distributed parameter beams were used in [2] to analyze a linearized system. Sturges [3] extended the work by Denavit and Hartenberg [4] to provide automated equation generation for tele-operators. Sturges considered only rigid members. His approach also utilized the 4×4 transformation matrix. Simunovic [5] also used this transformation concept to describe the variation of endpoint position of industrial manipulator arms in his analysis of their repeatability. The modal approach has been used to obtain dynamic equations, both linear and nonlinear with compliant members as in Hughes [6] for example. Such approaches are significantly more accurate when arm base and/or payload mass is not much greater than arm mass. They are more time consuming in both the modeling and simulation phase.

The plan of this paper is first to introduce the 4×4 transformation matrix and the particular form this matrix takes when describing beam deflection due to forces at the end of the beam. The position of the end of a chain of beams with deflection as shown in Fig. 1 is expressed as a product of transformation matrices. To obtain the chain compliance matrix the partial derivatives of this product are taken with respect to end point forces and moments. Then the equations of motion for the system are then developed, including the effect of feedback controlled joint angles between the beam segments.

The Transformation Matrix and Deformation

We are interested in a transformation between two coordinate systems whose origins are displaced from one another and whose axes are not parallel, as in Fig. 2. The position of point P is described in terms of coordinate system 2 by the vector $\mathbf{X}_2$. Given the vector $(\mathbf{X}_0)_1$ from 0_1 to 0_2 in terms of system 1 and the angles between the axes (or lines parallel to them), we desire to find the vector from 0_1 to P. This vector is easily found by the following matrix multiplication:

$$\begin{bmatrix} 1 \\ X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ (X_0)_1 & \cos(X_2, X_1) & \cos(Y_2, X_1) & \cos(Z_2, X_1) \\ (Y_0)_1 & \cos(X_2, Y_1) & \cos(Y_2, Y_1) & \cos(Z_2, Y_1) \\ (Z_0)_1 & \cos(X_2, Z_1) & \cos(Y_2, Z_1) & \cos(Z_2, Z_1) \end{bmatrix} \begin{bmatrix} 1 \\ X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \quad (1a)$$

or

$$\begin{bmatrix} 1 \\ -- \\ \mathbf{X}_1 \end{bmatrix} = \begin{bmatrix} 1 & \vdots & \mathbf{0}^T \\ \hline (\mathbf{X}_0)_1 & \vdots & \mathbf{R}_{21} \end{bmatrix} \begin{bmatrix} 1 \\ -- \\ \mathbf{X}_2 \end{bmatrix} \quad (1)$$

[1]Initial work was done under NASA contract NAS8-28055 [1] and was continued during the summer of 1976 as a faculty research fellow at NASA Johnson Space Center.

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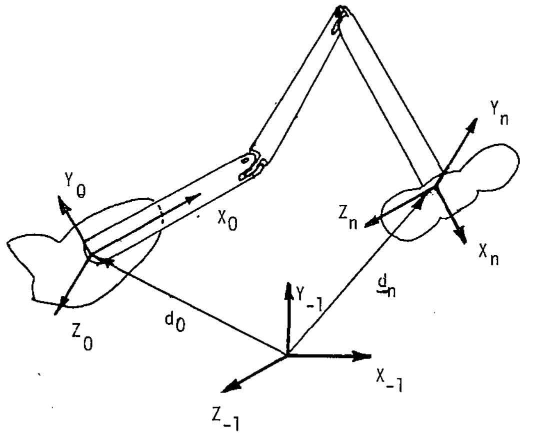

Fig. 1 Chain of elastic members with servo controlled joints

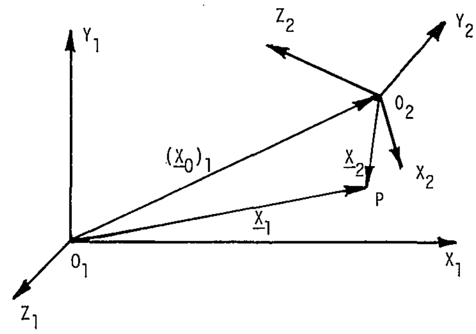

Fig. 2 A transformation between coordinates

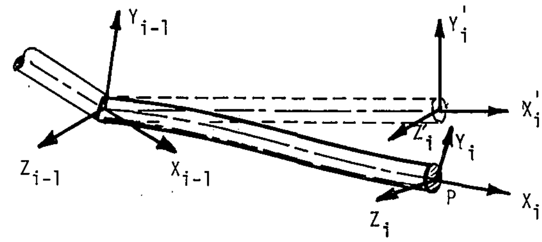

Fig. 3 Transformations between deformed and undeformed beams

The cosine terms are the cosines of the angles between intersecting lines parallel to the indicated axes. The sign convention is arbitrary for these angles since the cosine is an even function.

We are interested in coordinate transformations of two special types. One of these is the transformation due to undeformed joint angles and displacements. The other transformation is due to the deflection of arm segments under loading. The former has been described for both rotating and sliding joints by J. Denavit and R. S. Hartenberg [4][2] in terms of four independent parameters. The transformation due to deflection will be developed for the example of a simple beam under flexure, compression and torsion. Other elastic elements can be similarly derived.

The information we seek is the displacement and rotation of a jointed beam (a chain of beam segments jointed end-to-end) due to the application of loads. The position of the end of the beam can be described in a fixed reference coordinate system if one knows the transformation between the coordinate systems which are fixed to the individual segments. As seen in Fig. 3 the point P at the end of the beam can be described by two transformations, represented by two 4×4 matrices. The transformation $\mathbf{A}_i$ relates system i', the end point before deflection, to system $i - 1$. The transformation $\mathbf{E}_i$ relates system i to system i'.

$$\begin{bmatrix} 1 \\ ---- \\ \mathbf{X}_{i,i-1} \end{bmatrix} = \mathbf{A}_i\mathbf{E}_i \begin{bmatrix} 1 \\ ---- \\ \mathbf{X}_{i\ ,1} \end{bmatrix} = \mathbf{A}_i\mathbf{E}_i \begin{bmatrix} 1 \\ -- \\ \mathbf{0} \end{bmatrix} \qquad (2)$$

where: $\mathbf{X}_{i,i-1}$ = the position of the origin of system i in terms of system $i - 1$

$\mathbf{A}_i$ = transformation with no deflection

$\mathbf{E}_i$ = transformation due to deflection

$\mathbf{0}$ = a 3×1 vector whose elements are zero

$\mathbf{X}_{i,i}$ = location of point P in i coordinates. P is the origin of i in this case

[2]A reader consulting this paper should be aware of the fact that α_1 in [4] is defined with opposite sign convention to this paper and later papers by Denavit and Hartenberg.

Any number of these transformations may be combined by multiplying the transformation matrices. In terms of the reference system 0, the end of a beam with N joints is located at $\mathbf{X}_{N0}$ as is given by:

$$\begin{bmatrix} 1 \\ -- \\ \mathbf{X}_{N,0} \end{bmatrix} = \mathbf{A}_1\mathbf{E}_1\mathbf{A}_2 \ldots \mathbf{A}_i\mathbf{E}_i \ldots \mathbf{A}_N\mathbf{E}_N \begin{bmatrix} 1 \\ -- \\ \mathbf{0} \end{bmatrix} \qquad (3)$$

We would like the variation of this position vector due to applied forces and moments. First the elements of the $\mathbf{E}$ matrices must be found. For small deflections and small angles the elements of the matrix of equation (1) simplify as follows:

$$\mathbf{E}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \Delta X_i & 1 & \cos(90 + \theta_{Zi}) & \cos(90 - \theta_{Yi}) \\ \Delta Y_i & \cos(90 - \theta_{Zi}) & 1 & \cos(90 + \theta_{Xi}) \\ \Delta Z_i & \cos(90 + \theta_{Yi}) & \cos(90 - \theta_{Xi}) & 1 \end{bmatrix} \qquad (4)$$

where θ_{Xi}, θ_{Yi}, and θ_{Zi} are the angles of rotation about the X_i, Y_i, and Z_i axes, respectively. Using the fact that cos (90 deg + θ_{Zi}) = $-$ sin θ_{Zi}, and that for small angles sin $\theta_{Zi} \simeq \theta_{Zi}$, equation (4) simplifies to

$$\mathbf{E}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \Delta X_i & 1 & -\theta_{Zi} & \theta_{Yi} \\ \Delta Y_i & \theta_{Zi} & 1 & -\theta_{Xi} \\ \Delta Z_i & -\theta_{Yi} & \theta_{Xi} & 1 \end{bmatrix} \qquad (5)$$

This matrix can be expressed in terms of forces and moments on the segment. For a symmetrical beam the elements of this matrix are written in terms of influence coefficients. Assuming the neutral axis of the undeformed beam to lie along the X_{ii} axis,

$$\mathbf{E}_i = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \alpha_{Ci}F_{Xii} & 1 & -\alpha_{\theta Fi}F_{Yii} - \alpha_{\theta Mi}M_{Zii} & -\alpha_{\theta Fi}F_{Zii} + \alpha_{\theta Mi}M_{Yii} \\ \alpha_{XFi}F_{Yii} + \alpha_{XMi}M_{Zii} & \alpha_{\theta Fi}F_{Yii} + \alpha_{\theta Mi}M_{Zii} & 1 & -\alpha_{Ti}M_{Xii} \\ \alpha_{XFi}F_{Zii} - \alpha_{XMi}M_{Yii} & \alpha_{\theta Fi}F_{Zii} - \alpha_{\theta Mi}M_{Yii} & \alpha_{Ti}M_{Xii} & 1 \end{array}\right] \qquad (6)$$

where in general:

$F_{Xij}, F_{Yij}, F_{Zij}$ = forces at the end of beam i, in terms of coordinate system j. $\mathbf{F}_{ij}^T = [F_{Xij} \quad F_{Yij} \quad F_{Zij}]$

$M_{Xij}, M_{Yij}, M_{Zij}$ = moments at the end of beam i, in terms of coordinate system j. $\mathbf{M}_{ij}^T = [M_{Xij} \quad M_{Yij} \quad M_{Zij}]$

α_{Ci} = coefficient in compression, displacement/unit force

α_{XFi} = coefficient in bending, displacement/unit force

α_{XMi} = coefficient in bending, displacement/unit moment

$\alpha_{\theta Fi}$ = coefficient in bending, angle/unit force ($= \alpha_{XMi}$)

$\alpha_{\theta Mi}$ = coefficient in bending, angle/unit moment

α_T = coefficient in torsion, angle/unit moment

Now one must determine the forces $\mathbf{F}_{ii}$ and moments $\mathbf{M}_{ii}$ on segment i in coordinates i which result from the loads $\mathbf{F}_{N0}$ and $\mathbf{M}_{N0}$ on the end of the chain, segment N, in reference 0 coordinates. Equilibrium of the segments between the origin of coordinate system N and coordinate system i requires that

$$\begin{bmatrix} \mathbf{F}_{ii} \\ \hline \mathbf{M}_{ii} \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{R}_{0i} & \mathbf{0} \\ \hline \mathbf{r}_{ii} \times \mathbf{R}_{0i} & \mathbf{R}_{0i} \end{array}\right] \begin{bmatrix} \mathbf{F}_{N0} \\ \hline \mathbf{M}_{N0} \end{bmatrix} \tag{7}$$

where in general:

$\mathbf{R}_{ij}$ = rotation matrix from system i to system j

$\mathbf{r}_{ij}$ = distance vector from the origin of system i to the end of the chain (origin of system N) in terms of coordinates j

and "$\times$" indicates a vector cross product.

Compliance Matrix Evaluation. Having described the position of the end of the chain (after loading has been placed on the end) by the coordinate transformation, one could subtract from this vector the vector describing the position of the arm before loading as follows:

$$\Delta \mathbf{X} = [(\mathbf{A}_1\mathbf{E}_1\mathbf{A}_2\mathbf{E}_2 \ldots \mathbf{E}_{N-1}\mathbf{A}_N\mathbf{E}_N) - \mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_N] \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}.$$

Theoretically this would be correct. In practice the difference of these two vectors will be much smaller than the vectors themselves, leading to inaccuracies when the calculation is carried out with too few significant digits. A more accurate way is to evaluate the partial derivative of the position of the end with respect to end point loads. Consider for example, derivatives with respect to F_{XN0}. By the definition of $\mathbf{A}_i$ as the undeformed transformation

$$\frac{\partial \mathbf{A}_i}{\partial F_{XN0}} = \mathbf{0} \qquad i = 1, 2, \ldots, N$$

By the product rule one arrives at, in the example of F_{XN0},

$$\frac{\partial \mathbf{X}_{N0}}{\partial F_{XN0}} = \left[\sum_{i=1}^{N} \mathbf{A}_1\mathbf{E}_1 \ldots \mathbf{A}_1 \frac{\partial \mathbf{E}_i}{\partial F_{XN0}} \mathbf{A}_{i+1} \ldots \mathbf{A}_N\mathbf{E}_N\right] \begin{bmatrix} 1 \\ \hline \mathbf{0} \end{bmatrix}. \tag{8}$$

Similar expressions are available for the other force components' as well as for the moments.

For practical problems $\mathbf{E}_j$ is a matrix with ones on the diagonal and with off diagonal terms near zero. An accurate approximation is to replace it with the identity matrix in (8) except for $j = i$. In order to proceed one must evaluate

$$\frac{\partial \mathbf{E}_i}{\partial F_{XN0}}, \frac{\partial \mathbf{E}_i}{\partial F_{YN0}}, \frac{\partial \mathbf{E}_i}{\partial F_{ZN0}}, \frac{\partial \mathbf{E}_i}{\partial M_{XN0}}, \frac{\partial \mathbf{E}_i}{\partial M_{YN0}} \text{ and } \frac{\partial \mathbf{E}_i}{\partial M_{ZN0}}.$$

To do this one takes the derivative of the individual elements of (6) with respect to F_{XN0}, F_{YN0}, F_{ZN0}, M_{XN0}, M_{YN0}, and M_{ZN0}. This requires in turn that we evaluate the derivative of the forces and moments on the segments in local coordinates with respect to end point forces and moments, in reference zero coordinates. In vector form we desire

$$\frac{\partial \mathbf{F}_{ii}}{\partial \mathbf{F}_{N0}^T}, \frac{\partial \mathbf{F}_{ii}}{\partial \mathbf{M}_{N0}^T}, \frac{\partial \mathbf{M}_{ii}}{\partial \mathbf{F}_{N0}^T} \text{ and } \frac{\partial \mathbf{M}_{ii}}{\partial \mathbf{M}_{N0}^T}$$

Referring to (7) it is seen that these partial derivatives are readily evaluated if one assumes that $\mathbf{R}_{0i}$ and $\mathbf{r}_{ii} \times \mathbf{R}_{0i}$ are essentially independent of the loading, which is true to first order. The required derivatives are:

$$\begin{bmatrix} \dfrac{\partial}{\partial \mathbf{F}_{N0}} \\ \hline \dfrac{\partial}{\partial \mathbf{M}_{N0}} \end{bmatrix} [\mathbf{F}_{ii}^T \mid \mathbf{M}_{ii}^T] = \left[\begin{array}{c|c} \mathbf{R}_{0i} & \mathbf{r}_{ii} \times \mathbf{R}_{0i} \\ \hline \mathbf{0} & \mathbf{R}_{0i} \end{array}\right] \tag{9}$$

Displacement under loads is shown above to be the change in the vector $\mathbf{X}_{N0}$.

One also wishes to find the rotation of the end of the chain under loads. In order to obtain these results a rotation of coordinates is performed which produces a coordinate system N' which has the same origin as coordinates N but is parallel to the zero reference coordinates if there is no deflection. The transformation matrix is

$$A_0 = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{R}_{N'N} \end{array}\right],$$

where

$$\mathbf{R}^T_{N'N} = \prod_{i=0}^{N-1} \mathbf{R}_{i+1,i}.$$

The matrix $\mathbf{R}^T_{N'N}$ will have been evaluated in the course of computing (8).

The derivative with respect to a force or moment, F_{XN0} for example is now

$$\frac{\partial}{\partial F_{XN0}} \begin{bmatrix} 1 \\ \hline \mathbf{X}_{N0} \end{bmatrix} = \frac{\partial}{\partial F_{XN0}} \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline (\mathbf{X}_0)_{N0} & \mathbf{R}_{N'0} \end{array}\right] \begin{bmatrix} 1 \\ \hline \mathbf{0} \end{bmatrix}$$

The derivative of the rotation submatrix can be simplified in a manner similar to the rotation submatrix of $\mathbf{E}_i$ in (4) and (5). The derivative of that submatrix then yields the rotations

$$\frac{\partial}{\partial F_{XN0}} \mathbf{R}_{N'0} = \frac{\partial}{\partial F_{XN0}} \begin{bmatrix} 1 & -\theta_{ZN0} & \theta_{YN0} \\ \theta_{ZN0} & 1 & -\theta_{XN0} \\ -\theta_{YN0} & \theta_{XN0} & 1 \end{bmatrix}$$

As a result of the preceding development we write

$$\frac{\partial}{\partial F_{XN0}}\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline (X_0)_{N0} & 1 & -\theta_{ZN0} & \theta_{YN0} \\ (Y_0)_{N0} & \theta_{ZN0} & 1 & -\theta_{XN0} \\ (Z_0)_{N0} & -\theta_{YN0} & \theta_{XN0} & 1 \end{array}\right]$$

$$= \sum_{i=1}^{N} \mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_i \frac{\partial}{\partial F_{XN0}} \mathbf{E}_i\mathbf{A}_{i+1} \ldots \mathbf{A}_N\mathbf{A}_0 \qquad (10)$$

Similar expressions can be written for derivatives with respect to the other five components of moment and force. The $\mathbf{A}_i$ terms depend on the undeformed geometry. $\mathbf{E}_i$ is written in (6) in terms of influence coefficients of the deformable members and forces on the member in the local coordinates of that member. Its derivative with respect to end point force required an expression for the variation of forces on the member in terms of the end point forces. This expression (9) was obtained from the geometry using transformation matrices.

Equation (10) and its five companions allow one to write the 6×6 compliance matrix $\mathbf{C}$ as

$$\mathbf{C} = \begin{bmatrix} \dfrac{\partial}{\partial \mathbf{F}_{N0}} \\ \dfrac{\partial}{\partial \mathbf{M}_{N0}} \end{bmatrix} [\mathbf{X}_{N0}{}^T \quad \boldsymbol{\theta}_{N0}{}^T].$$

$\mathbf{C}$ is symmetric and nonsingular for physical systems. Its inverse is the spring constant matrix

$$\mathbf{K}_s = \mathbf{C}^{-1}.$$

Equations of Small Motion

Consider the equations of small motion for a chain of massless beam segments with a mass attached at each end, centers of mass at the origins of coordinate systems 0 and n as in Fig. 1. The positions and angles which locate and orient these masses with respect to an inertial reference frame -1 are contained in the vectors $\mathbf{Z}_0$ and $\mathbf{Z}_n$, respectively.

$$\mathbf{Z}_0 = [\mathbf{d}_0{}^T \quad \boldsymbol{\phi}_0{}^T]^T; \qquad \mathbf{Z}_n = [\mathbf{d}_n{}^T \quad \boldsymbol{\phi}_n{}^T]^T$$

$\mathbf{d}_0$, $\mathbf{d}_n$—position vector from the origin of -1 to the origins of 0 and n, respectively.

$\boldsymbol{\phi}_0$, $\boldsymbol{\phi}_n$—vector of small rotations of mass 0 and n, respectively, from an initial undeformed orientation.

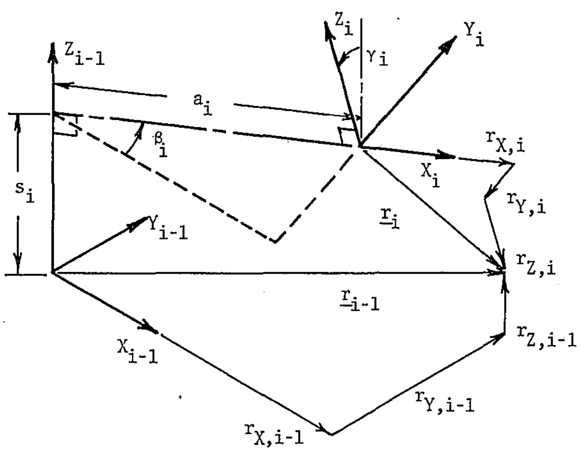

Fig. 4 Transformation for joints

When the beam segments of the chain are rigidly connected to one another the compliance matrix $\mathbf{C}$ or its inverse $\mathbf{K}_s$ and the inertia matrices of mass 0, $\mathbf{J}_0$, and mass n, $\mathbf{J}_n$, are sufficient to describe the natural or unforced motions of the system. Upon the inclusion of servo controlled actuators at the joints additional complexities arise which are resolved here.

The cases of controlled joints will be considered, both with rotary motion. Motion at the joints means a parameter in the transformation matrix will vary. With certain constraints on the way consecutive axes are arranged one can describe any transformation with four parameters, two distances and two angles, as illustrated in Fig. 4. This special form will be used for the $\mathbf{A}_i$ matrices defining the undeformed relative position and angle of the consecutive segments.

$$\mathbf{A}_i = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ a_i \cos\beta_i & \cos\beta_i & -\sin\beta_i\cos\gamma_i & \sin\beta_i\sin\gamma_i \\ a_i \cos\beta_i & \sin\beta_i & \cos\beta_i\cos\gamma_i & -\cos\beta_i\sin\gamma_i \\ s_i & 0 & \sin\gamma_i & \cos\gamma_i \end{array}\right]$$

$$\begin{bmatrix} 1 \\ -- \\ \mathbf{r}_{i-1} \end{bmatrix} = \mathbf{A}_i \begin{bmatrix} 1 \\ -- \\ \mathbf{r}_i \end{bmatrix}$$

Axis X_i must be perpendicular to axis Z_{i-1}

a_i = length of the perpendicular from Z_{i-1} to X_i

γ_i = angle from positive Z_{i-1} to positive Z_i, measured counter clockwise about positive X_i

β_i = angle from positive X_{i-1} to positive X_i measured counter clockwise about positive X_i

s_i = distance along Z_{i-1} from X_{i-1} to X_i. Takes sign from orientation of positive Z_{i-1}.

The case of γ_i or β_i as the variable parameter will be allowed.

Consider a general joint angle position and velocity feedback law controlling the joints in which

$$\text{Torque by Actuators} = \mathbf{K}_c(\boldsymbol{\theta}_J - \boldsymbol{\theta}_{dJ}) + \mathbf{D}_c(\dot{\boldsymbol{\theta}}_J - \dot{\boldsymbol{\theta}}_{dJ})$$

$\boldsymbol{\theta}_J$ = vector of m controlled joint angles measured from an arbitrary reference point

$\boldsymbol{\theta}_{dJ}$ = vector of m desired joint angles

$\dot{\boldsymbol{\theta}}_J$ = vector of m controlled joint angular velocities

$\dot{\boldsymbol{\theta}}_{dJ}$ = vector of m desired joint angular velocities

$\mathbf{K}_c$ = $m \times m$ matrix of position control gains

$\mathbf{D}_c$ = $m \times m$ matrix of velocity control gains

Actuator torques can be expressed in terms of structural deflections as well. Consider 0 fixed and move n slightly from its equilibrium position $\mathbf{Z}_{ne}$ to a new position $\mathbf{Z}_n$. The vector of forces and moments on mass n may be determined from a spring constant matrix $\mathbf{K}_s$ as

$$\begin{bmatrix} \text{Forces} \\ \text{Moments} \end{bmatrix}_{\substack{\text{on mass at } n \\ n \text{ displaced}}} = -\mathbf{K}_s(\mathbf{Z}_n - \mathbf{Z}_{ne})$$

The resulting torques on the actuators are linearly related to these forces and moments by the matrix $\mathbf{H}$.

$$\begin{bmatrix} \text{Torques on actuators} \\ \text{with } n \text{ displaced} \end{bmatrix} = \mathbf{H}\mathbf{K}_s(\mathbf{Z}_n - \mathbf{Z}_{ne})$$

To determine $\mathbf{H}$ one needs a matrix previously evaluated now called $\mathbf{XCR}_h$

$$\mathbf{XCR}_h = \mathbf{r}_{hh} \times \mathbf{R}_{0h}$$

and the matrix $\mathbf{R}_{0h}$ when joint h is a controlled joint. $\mathbf{XCR}_h$ gives the appropriate moment arms which relate end point force to torque at the joint h. $\mathbf{R}_{0h}$ rotates coordinates appropriately to relate end point moments to actuator torque. For joint h as the ith controlled joint, joints ordered from 0 to n, we have

$$
\begin{aligned}
\mathbf{H}(i,j) &= \text{element in } i\text{th row, } j\text{th column of } \mathbf{H} \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant 6 \\
\mathbf{H}(i,j) &= \mathbf{XCR}_h(k,j), \quad 1 \leqslant j \leqslant 3 \quad 1 \leqslant i \leqslant m \\
\mathbf{H}(i,j) &= \mathbf{R}_{0h}(k, j-3), \quad 3 \leqslant j \leqslant 6 \quad 1 \leqslant i \leqslant m \\
k &= 1 \text{ if } \gamma_h \text{ is the } i\text{th element of } \boldsymbol{\theta}_J \\
k &= 3 \text{ if } \beta_h \text{ is the } i\text{th element of } \boldsymbol{\theta}_J \\
m &= \text{total number of controlled joints}
\end{aligned}
$$

If mass 0 is displaced slightly one can determine the torques on the actuators by first determining the equivalent motion of mass n. For small motions

$$
\begin{aligned}
(\text{Equivalent movement of } n) &= -\left[\begin{array}{c:c} \mathbf{I} & \mathbf{XC}^T \\ \hdashline \mathbf{0} & \mathbf{I} \end{array}\right](\mathbf{Z}_0 - \mathbf{Z}_{0e}) \\
&= \mathbf{T}^T(\mathbf{Z}_0 - \mathbf{Z}_{0e})
\end{aligned}
$$

The matrix XC effects a cross product of the vector length of the chain $\mathbf{r}_{0n}$ with the vector it is multiplied by

$$
\mathbf{XC} = \begin{bmatrix} 0 & -r_{Z0n} & r_{Y0n} \\ r_{Z0n} & 0 & -r_{X0n} \\ -r_{Y0n} & r_{X0n} & 0 \end{bmatrix}
$$

The matrix $\mathbf{XC}^T$ gives the effect of rotations at 0 on equivalent displacements at n.

The total torque on the actuators is

$$
\begin{aligned}
(\text{Torque on Actuators}) &= \mathbf{HK}_s\{(\mathbf{Z}_n - \mathbf{Z}_{ne}) - \mathbf{T}^T(\mathbf{Z}_0 - \mathbf{Z}_{0e})\} \\
&= \mathbf{HK}_s\{\mathbf{Z}_n - \mathbf{T}^T\mathbf{Z}_0 - (\mathbf{Z}_{ne} - \mathbf{T}^T\mathbf{Z}_{0e})\}
\end{aligned}
$$

Because $\mathbf{XC}$ is skew symmetric, $\mathbf{T}$ is orthogonal, i.e.

$$
\begin{aligned}
\mathbf{XC}^T &= -\mathbf{XC} \\
\mathbf{T}^T\mathbf{T} &= \mathbf{I}
\end{aligned}
$$

The actuator torque is proportional to the difference of the position vectors of the two bodies, as measured from an equilibrium configuration. The vector $(\mathbf{Z}_{ne} - \mathbf{T}^T\mathbf{Z}_{0e})$ which determines the equilibrium position depends only on the joint positions $\boldsymbol{\theta}_J$ and for small variations of $\boldsymbol{\theta}_J$ from zero the relation is linear given by the matrix G.

$$
(\mathbf{Z}_{ne} - \mathbf{T}^T\mathbf{Z}_{0e}) = \mathbf{G}\boldsymbol{\theta}_J
$$

If one equates the two expressions for actuator torques and solves for $\dot{\boldsymbol{\theta}}_J$

$$
\dot{\boldsymbol{\theta}}_J = \mathbf{D}_c^{-1}\{(-\mathbf{HK}_s\mathbf{G} - \mathbf{K}_c)\boldsymbol{\theta}_J + \mathbf{HK}_s(\mathbf{Z}_n - \mathbf{T}^T\mathbf{Z}_0) + \mathbf{K}_c\boldsymbol{\theta}_{dJ}\} + \dot{\boldsymbol{\theta}}_{dJ} \tag{11}
$$

The matrix $\mathbf{G}$ yields the incremental change in the position of the body at n for variations in $\boldsymbol{\theta}_J$ when 0 is held fixed. The matrix $\mathbf{H}$ yields the equilibrium forces and/or torques on the joint for forces and/or torques applied by body n. By the principle of virtual work $\mathbf{G} = \mathbf{H}^T$.

Equation (11) is in state variable form if one considers as state variables $\boldsymbol{\theta}_J$, $\mathbf{Z}_n$, $\mathbf{Z}_0$ and $\boldsymbol{\theta}_{dJ}$ and considers $\boldsymbol{\theta}_{dJ}$ as an input.

The remaining equations of motion are found by summing moments and torques on the center of mass of the rigid masses. For small motions

$$
\frac{d}{dt}(\dot{\mathbf{Z}}_n) = -\mathbf{J}_n^{-1}\mathbf{K}_s(\mathbf{Z}_n - \mathbf{T}^T\mathbf{Z}_0 - \mathbf{G}\boldsymbol{\theta}_J) \tag{12}
$$

$$
\frac{d}{dt}(\mathbf{Z}_n) = \dot{\mathbf{Z}}_n \tag{13}
$$

and similarly if one considers mass 0 with resultant external forces $\mathbf{F}_0$

$$
\frac{d}{dt}(\dot{\mathbf{Z}}_0) = \mathbf{J}_0^{-1}\mathbf{TK}_s(\mathbf{Z}_n - \mathbf{T}^T\mathbf{Z}_0 - \mathbf{G}\boldsymbol{\theta}_J) + \mathbf{J}_0^{-1}\mathbf{B}_T\mathbf{F}_0 \tag{14}
$$

$$
\frac{d}{dt}(\mathbf{Z}_0) = \dot{\mathbf{Z}}_0 \tag{15}
$$

where $\mathbf{B}_T$ resolves the resultant force into forces through the center of gravity of the inertia at 0 and moments about the center of gravity.

$\mathbf{J}_0$ and $\mathbf{J}_n$ are inertia matrices about the centers of gravity of the masses of 0 and n, respectively.

When combined in matrix form, equations (11)–(15) yield the usual state variable form for the 24 + 2m order system.

If one is interested in the apparent relative position of n as viewed from 0, introduce new state variables equal to $\mathbf{Z}_n - \mathbf{T}^T\mathbf{Z}_0$, and its derivative. A convenient definition of parameters is

$$
\begin{aligned}
\mathbf{J}_{0T} &= \mathbf{T}^{-1}\mathbf{J}_0(\mathbf{T}^{-1})^T = \mathbf{T}^T\mathbf{J}_0\mathbf{T} \\
\mathbf{J}_{0T}^{-1} &= \mathbf{T}^T\mathbf{J}_0^{-1}\mathbf{T}
\end{aligned}
$$

The set of state variables can be completed with $\mathbf{J}_{0T}^{-1}\mathbf{J}_n\mathbf{Z}_n + \mathbf{T}^T\mathbf{Z}_0$ and the derivative, $\mathbf{J}_{0T}^{-1}\mathbf{J}_n\dot{\mathbf{Z}}_n + \mathbf{T}^T\dot{\mathbf{Z}}_0$. The total state vector using these state variables becomes

$$
\begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_1 \\ \mathbf{W}_2 \\ \dot{\mathbf{W}}_2 \\ \boldsymbol{\theta}_J \\ \boldsymbol{\theta}_{dJ} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{0T}^{-1}\mathbf{J}_n\mathbf{Z}_n + \mathbf{T}^T\mathbf{Z}_0 \\ \mathbf{J}_{0T}^{-1}\mathbf{J}_n\dot{\mathbf{Z}}_n + \mathbf{T}^T\dot{\mathbf{Z}}_0 \\ \mathbf{Z}_n - \mathbf{T}^T\mathbf{Z}_0 \\ \dot{\mathbf{Z}}_n - \mathbf{T}^T\dot{\mathbf{Z}}_0 \\ \boldsymbol{\theta}_J \\ \boldsymbol{\theta}_{dJ} \end{bmatrix}
$$

The equations of motion in state variables form become

$$
\frac{d}{dt}\begin{bmatrix} \mathbf{W}_1 \\ \dot{\mathbf{W}}_1 \\ \mathbf{W}_2 \\ \dot{\mathbf{W}}_2 \\ \boldsymbol{\theta}_J \\ \boldsymbol{\theta}_{dJ} \end{bmatrix} = \left[\begin{array}{c:c:c:c:c:c}
\mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -(\mathbf{J}_{0T}^{-1} + \mathbf{J}_n^{-1})\mathbf{K}_s & \mathbf{0} & (\mathbf{J}_{0T}^{-1} + \mathbf{J}_n^{-1})\mathbf{K}_s\mathbf{H}^T & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{D}^{-1}\mathbf{HK}_s & \mathbf{0} & -\mathbf{D}_c^{-1}(\mathbf{HK}_s\mathbf{H}^T + \mathbf{K}_c) & \mathbf{D}^{-1}\mathbf{K}_c \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right]\begin{bmatrix} \mathbf{W}_1 \\ \dot{\mathbf{W}}_1 \\ \mathbf{W}_2 \\ \dot{\mathbf{W}}_2 \\ \boldsymbol{\theta}_J \\ \boldsymbol{\theta}_{dJ} \end{bmatrix} + \left[\begin{array}{c:c}
\mathbf{0} & \mathbf{0} \\
\mathbf{J}_{0T}^{-1}\mathbf{T}^T\mathbf{B}_T\mathbf{T} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} \\
-\mathbf{J}_{0T}^{-1}\mathbf{T}^T\mathbf{B}_T\mathbf{T} & \mathbf{0} \\
\mathbf{0} & \mathbf{I} \\
\mathbf{0} & \mathbf{I}
\end{array}\right]\begin{bmatrix} \mathbf{F}_0 \\ \dot{\boldsymbol{\theta}}_{dJ} \end{bmatrix} \tag{16}
$$

The equations for $\mathbf{W}_1$ and $\dot{\mathbf{W}}_1$ are not coupled to the remaining equations and represent the motion of the system as a whole. The remaining equations represent the relative motion of the two masses and may be solved independently from the overall motion. If the joints are locked $\boldsymbol{\theta}_J = \boldsymbol{\theta}_{dJ} = \mathbf{0}$, the relevant equation for relative motion is

$$
\frac{d}{dt}\begin{bmatrix} \mathbf{W}_2 \\ \dot{\mathbf{W}}_2 \end{bmatrix} = \left[\begin{array}{c:c} \mathbf{0} & \mathbf{I} \\ -(\mathbf{J}_{0T}^{-1} + \mathbf{J}_n^{-1})\mathbf{K}_s & \mathbf{0} \end{array}\right]\begin{bmatrix} \mathbf{W}_2 \\ \dot{\mathbf{W}}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{J}_{0T}^{-1}\mathbf{T}^T\mathbf{B}_T\mathbf{T}\mathbf{F}_0 \end{bmatrix}
$$

Summary

The linear equations of spatial motion were developed for two rigid masses connected by a chain of massless beam segments with an arbitrary number of controlled joint rotations. This was facilitated by the technique developed using transformation matrices to describe the beam deformation and the position and orientation of consecutive beam segments. These techniques are adaptable to computer program implementation. They have been so implemented and allow a very efficient determination of the equations of motion for such systems.

This paper described the case of two masses and symmetrical beam segments. The method can be extended to additional masses without difficulty. Any elastic elements for which influence coefficients can be determined can be used as the segments of the chain. In addition the technique has been used to model systems with parallel structural elements such as drive trains [1].

The method has been used at NASA's Johnson Space Center for modeling the dynamics of the Space Shuttle Orbiter, the Orbiter's Remote Manipulator System and the large payloads of the Orbiter. It provided an efficient approximation to the system equations and is complementary to the more complete models which include the effects of distributed mass and/or nonlinearities. Projected space missions involve large cranes constructing large space structures. The techniques described here can be effectively employed in the analysis of such systems.

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